On Special Values of Zeta Functions Associated with a Self-Dual Cone (P-Adic L-Functions and Algebraic Number Theory)

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以下に掲げるのは松島与三氏還暦記念論文集(Birkhauser)のための原稿の一部である。京都の研究集会ではこの後半についてお話ししたので、その要約を提出する予定であったが、都合上原稿(の原稿)のまゝ出させて頂くことにした。本文で説明した通り、こゝに述べる方法は本質的に故新谷氏[11]のアイディアによるものである。$r=2$ (circular cone)の場合にはより精密な計算をすることができ、栗原氏も独立に結果を得ておられるが、これについてはまた別の機会に触れたいと思う。

§1. Introduction

To explain the main idea of this paper, and also to fix some notations, we start with reviewing the classical case of Riemann zeta function. As usual, we set

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (\mathrm{Re}\ s>1),$$

$$\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}dx \qquad (\mathrm{Re}\ s>0).$$

Then, for $\mathrm{Re}\ s>1$, one obtains

$$\begin{aligned}\Gamma(s)\zeta(s) &= \sum_{n=1}^{\infty} n^{-s} \int_0^{\infty} x^{s-1} e^{-x}\, dx \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} x'^{s-1} e^{-nx'}\, dx' \qquad (x = nx') \\ &= \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx .\end{aligned}$$

We put

$$b(x, y) = \frac{e^{xy}}{e^x - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu(y)}{\nu!} x^{\nu-1} \qquad (|x|<2\pi),$$

where

$$B_\nu(y) = \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} b_\mu\, y^{\nu-\mu}$$

is the Bernoulli polynomial, in which the b_μ are the Bernoulli numbers:

$$b_0 = 1, \quad b_1 = -\frac{1}{2},$$

$$b_\nu = \begin{cases} (-1)^{\frac{\nu}{2}-1} B_{\frac{\nu}{2}} & (\nu \text{ even}, \geqslant 2), \\ 0 & (\nu \text{ odd}, \geqslant 3). \end{cases}$$

Then the above integral can be transformed into a contour integral of the form

$$(1.1) \qquad \Gamma(s)\zeta(s) = (e^{2\pi is} - 1)^{-1} \int_{I(\varepsilon,\infty)} x^{s-1}\, b(x, 0)\, dx,$$

where $I(\varepsilon,\infty)$ denotes the contour consisting of the half-line $[\varepsilon, \infty)$ taken twice in opposite directions and of a (small) circle of radius ε

about the origin taken in the counterclockwise direction. The contour integral is absolutely convergent for all $s \in \mathbf{C}$, so that the function $\Gamma(s)\zeta(s)$ can be analytically continued to a meromorphic function on $\mathbf{C}$. Moreover, in virtue of the functional equation of the gamma function:

$$(1.2) \qquad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} = 2\pi i \, \frac{e^{\pi i s}}{e^{2\pi i s} - 1},$$

one obtains

$$(1.3) \qquad \zeta(s) = e^{-\pi i s}\, \Gamma(1 - s) \cdot \frac{1}{2\pi i} \int_{I(\varepsilon, \infty)} x^{s-1}\, b(x, 0)\, dx.$$

This shows that $\zeta(s)$ is holomorphic for $\mathrm{Re}\, s < 1$. In particular, for $s = 1 - m$, $m \in \mathbf{Z}^+$ (positive integers), the contour integral reduces to the residue of $x^{-m} b(x, 0)$ at $x = 0$, i.e., $b_m/m!$. Hence one obtains

$$(1.4) \qquad \zeta(1 - m) = (-1)^{m-1}(m-1)!\, \frac{b_m}{m!} = (-1)^{m-1}\frac{b_m}{m}.$$

Thus $\zeta(1 - m)$ $(m \in \mathbf{Z}^+)$ is rational. In particular,

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12},$$

$$\zeta(-2\mu) = 0, \quad \zeta(1 - 2\mu) = (-1)^{\mu}\frac{B_\mu}{2\mu} \qquad (\mu \geqslant 1).$$

This result has been generalized by Hecke, Klingen and Siegel [13] to the case of Dedekind zeta functions of totally real number fields. More recently, Shintani [11] gave a proof based on a direct generaliztion of the classical method explained above. Zeta functions attached to self-dual homogeneous cones have been studied by Siegel [13] in a special case of quadratic cones, and by Sato-Shintani [8] in a more general context of "prehomogeneous spaces". (Cf. also Shintani [9], [10].) On the other hand, the gamma functions attached to self-dual homogeneous cones were studied by Koecher [5], Gindikin [3] and others (cf. e.g., Resnikoff [6]). In this paper, we try to extend Shintani's method (i.e., the classical method) to examine the rationality of the special values of zeta functions attached to self-dual homogeneous cones.

§2. The gamma function of a self-dual homogeneous cone

2.1. Let U be a real vector space of dimension n, endowed with a positive definite inner product $\langle\ \rangle$. By a "cone" in U we always mean a non-degenerate open convex cone in U with vertex at the origin, i.e., a non-empty open set $\mathcal{R}$ in U such that

$$x, y \in \mathcal{R}, \quad \lambda, \mu \in \mathbb{R}^+ \Longrightarrow \lambda x + \mu y \in \mathcal{R}$$

and such that $\mathcal{R}$ does not contain any straight line. A cone $\mathcal{R}$ in U is called <u>homogeneous</u> if the group of linear automorphisms

$$G(\mathcal{R}) = \{ g \in GL(U) \mid g(\mathcal{R}) = \mathcal{R} \}$$

is transitive on $\mathcal{R}$; and $\mathcal{R}$ is called <u>self-dual</u> if the "dual" of

$$\mathcal{R}^* = \{ x \in U \mid \langle x, y \rangle > 0 \ \text{ for all } \ y \in \bar{\mathcal{R}} - \{0\} \}$$

coincides with $\mathcal{R}$.

Let $\mathcal{R}$ be a self-dual homogeneous cone in U and $G = G(\mathcal{R})^{\circ}$. Then it is well-known (e.g., Satake [7]) that the Zariski closure of G (in $GL(U)$) is a reductive algebraic group, containing $G(\mathcal{R})$ as a subgroup of finite index, and $g \longmapsto {}^t g^{-1}$ is a Cartan involution of G; the corresponding maximal compact subgroup $K = G \cap O(U)$ coincides with the isotropy subgroup of G at a "base point" $e \in \mathcal{R}$ (which is not unique, but will be fixed once and for all). Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the corresponding Cartan decomposition of $\mathfrak{g}$ = Lie G. Then $\mathfrak{k}$ = Lie K and one has for $T \in \mathfrak{g}$

(2.1) $$T \in \mathfrak{k} \iff {}^tT = -T \iff Te = 0.$$

It follows that, for each $u \in U$, there exists a uniquely determined element $T_u \in \mathfrak{p}$ such that $T_u e = u$. It is well-known that the vector space U endowed with a product

$$u \circ u' = T_u u' \qquad (u, u' \in U)$$

becomes a formally real Jordan algebra (cf. Braun-Koecher [2], or Satake [7]).

We define the (regular) <u>trace</u> on U by

(2.2) $$\tau(u) = \mathrm{tr}(T_u).$$

For the given $(\mathcal{D}, e)$, one may assume (by Schur's lemma) that the inner product $<\ >$ is so normalized that one has

(2.3) $$< u, u' > = \tau(u \circ u') \qquad (u, u' \in U).$$

Next, let $u \in \mathcal{D}$. Then, since G is transitive on $\mathcal{D}$, there exists $g_1 \in G$ such that $u = g_1 e$. We define the (regular) norm $N(u)$ by

$$N(u) = \det(g_1),$$

which is clearly independent of the choice of g_1. There exists a unique element $u_1 \in U$ such that $u = \exp u_1$ (which is defined to be $(\exp T_{u_1})e$); then by definition one has

(2.4) $$N(u) = \det(\exp T_{u_1}) = e^{\tau(u_1)}.$$

In terms of the "quadratic multiplication" $P(u) = 2\,T_u^{\,2} - T_{u^2}$, one can also write $N(u) = \det(P(u))^{\frac{1}{2}}$. By the definition, it is clear that

(2.5) $$N(e) = 1, \quad N(gu) = \det(g)\,N(u) \quad (g \in G(\mathcal{D}),\ u \in \mathcal{D}),$$

which characterizes the norm uniquely. Denoting the Euclidean measue on U by du, we see that $d_{\mathcal{D}}(u) = N(u)^{-1}du$ is an invariant measure on $\mathcal{D}$.

<u>Example</u>. Let $U = \mathrm{Sym}_r(\mathbb{R})$ (the space of real symmetric matrices of degree r) and $\mathcal{D} = \mathcal{P}_r(\mathbb{R})$ (the cone of positive definite elements in U). Then one has

$$T_u(u') = \frac{1}{2}(uu' + u'u)$$

and so

$$\tau(u) = \frac{r+1}{2}\,\mathrm{tr}(u), \qquad N(u) = \det(u)^{\frac{r+1}{2}}.$$

2.2. We define the gamma function of the cone Ω by

$$\Gamma_\Omega(s) = \int_\Omega N(u)^{s-1} e^{-\tau(u)}\, du \tag{2.6}$$

which converges absolutely for Re s sufficiently large (actually for Re s $> 1 - \frac{r}{n}$ as we will see later).

LEMMA 2.1. Suppose that the inner product $\langle\ \rangle$ is normalized by (2.3). Then one has for any $v \in \Omega$

$$\int_\Omega N(u)^{s-1} e^{-\langle u, v\rangle} du = \Gamma_\Omega(s)\, N(v)^{-s}. \tag{2.7}$$

Proof. Let $v = g_1 e$ with $g_1 \in G$ and put $u' = {}^t g_1 u$. Then one has

$$\langle u, v\rangle = \langle u, g_1 e\rangle = \langle u', e\rangle = \tau(u').$$

Hence by (2.5) the left-hand side of (2.7) is equal to

$$\int_\Omega N(u)^s e^{-\langle u, v\rangle} d_\Omega(u)$$

$$= \int_\Omega (\det(g_1)^{-1} N(u'))^s e^{-\tau(u')} d_\Omega(u')$$

$$= N(v)^{-s}\, \Gamma_\Omega(s), \text{ q.e.d.}$$

It is known that the function $\Gamma_\Omega(s)$ can be expressed as a product of ordinary gamma functions (cf. e.g., Resnikoff loc. cit.). For the sake of completeness, we sketch a proof . First, it is clear that, if

$$\Omega = \Omega_1 \times \cdots \times \Omega_m$$

is the decomposition of Ω into the direct product of irreducible (self-dual homogeneous) cones, then one has

$$\Gamma_\Omega(s) = \Gamma_{\Omega_1}(s) \cdots \Gamma_{\Omega_m}(s).$$

Hence, for our purpose, we may assume that Ω is irreducible.

We need the root structure of $\mathfrak{g}$, which can be determined as follows. Let

$$e = \sum_{i=1}^{r} e_i, \qquad e_i e_j = \delta_{ij}\, e_i \tag{2.8}$$

be a decomposition of e (in the Jordan algebra U) into the sum of mutually orthogonal primitive idempotents. ("Primitive" means that each e_i can not be decomposed into the sum of mutually orthogonal idempotents any more.) Then we obtain the following decomposition of U into the direct sum of subspaces ("Peirce decomposition").

(2.9) $$U = \bigoplus_{1 \leq i \leq j \leq r} U_{ij} ,$$

where

$$U_{ii} = \left\{ u \in U \,\middle|\, e_i u = u \right\},$$

$$U_{ij} = \left\{ u \in U \,\middle|\, e_i u = e_j u = \tfrac{1}{2} u \right\} \qquad (i \neq j).$$

Then one has $e_k u = 0$ for $u \in U_{ij}$, $k \neq i, j$. Moreover

(2.10) $$\dim U_{ii} = 1, \quad \dim U_{ij} = d \quad (i \neq j),$$

where d is a positive integer depending on the irreducible cone $\mathcal{R}$. (For instance, one has $d = 1$ for $\mathcal{R} = \mathcal{P}_r(R)$.) From (2.9), (2.10) one has the relation

(2.11) $$n = r + \frac{1}{2} r(r-1)d, \qquad \text{i.e.,} \qquad d = \frac{2(n-r)}{r(r-1)} .$$

It follows that

(2.12) $$\tau(e_i) = \mathrm{tr}(T_{e_i}) = 1 + \frac{1}{2}(r-1)d = \frac{n}{r} .$$

Put

(2.13) $$\mathfrak{a} = \left\{ T_{e_i} \quad (1 \leq i \leq r) \right\} .$$

Then $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$ of dimension r contained in $\mathfrak{p}$. We denote by (λ_i) the basis of $\mathfrak{a}^*$ (the dual space of $\mathfrak{a}$) dual to (T_{e_i}), i.e., one has the relation

$$T = \sum_{i=1}^{r} \lambda_i(T)\, T_{e_i} \qquad (T \in \mathfrak{a}).$$

We put $\alpha_{ij} = \frac{1}{2}(\lambda_i - \lambda_j)$ $(i \neq j)$.

PROPOSITION 1. The root system of $\mathfrak{g}$ relative to $\mathfrak{a}$ is given by $\Phi = \{\alpha_{ij}\ (i \neq j)\}$. The root space $\mathfrak{g}(\alpha_{ij})$ corresponding to α_{ij} is given by

$$\mathfrak{g}(\alpha_{ij}) = \{T_u + [T_{e_i - e_j}, T_u] \mid u \in U_{ij}\}. \tag{2.14}$$

This can be verified by a straightforward computation; see e.g., Ash et al. [1] Ch. II, §3. Proposition 1 implies that the R-rank of $\mathfrak{g}$ is equal to r and the root system Φ is of type (A_{r-1}).

2.3. Next we determine the Haar measure of G. Put

$$\mathfrak{n} = \sum_{i<j} \mathfrak{g}(\alpha_{ij})$$

and let A, N be the analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{n}$, respectively. Then one has an Iwasawa decomposition $G = NA \cdot K (\approx N \times A \times K)$, which gives rise to the following formula for (the volume element of) a (biinvariant) Haar measure on G:

$$dg = c_1\, e^{-2\rho(\log a)} d\underline{n}\, da\, dk \tag{2.15}$$

for $g = \underline{n}ak$ with $\underline{n} \in N$, $a \in A$, $k \in K$, where $d\underline{n}$, da, dk denote Haar measures on N, A, K, respectively, c_1 is a positive constant depending on the normalization of the Haar measures, and ρ is a linear form on $\mathfrak{a}$ defined by

$$\rho(T) = \frac{1}{2} \operatorname{tr}(\operatorname{ad} T \mid \mathfrak{n}) \qquad (T \in \mathfrak{a});$$

by Proposition 1 one has

$$\rho = \frac{d}{2} \sum_{i<j} \alpha_{ij} = \frac{d}{2} \sum_{i=1}^{r} (r - 2i + 1)\lambda_i. \tag{2.16}$$

The Haar measure of K is always normalized by $\int_K dk = 1$. We make an identification $A = (\mathbb{R}^+)^r$ by the correspondence $a \longleftrightarrow (t_i)$ defined by the relation $a = \exp(\sum \lambda_i T_{e_i})$, $t_i = e^{\lambda_i}$; then one has $da = \prod (dt_i/t_i)$. Moreover one has

$$\begin{aligned} \det(a) &= e^{\tau(\sum \lambda_i e_i)} = e^{\frac{n}{r}\sum \lambda_i} = \Big(\prod_{i=1}^{r} t_i\Big)^{\frac{n}{r}}, \\ a \cdot e &= \sum e^{\lambda_i} e_i = \sum_{i=1}^{r} t_i e_i, \end{aligned} \tag{2.17}$$

$$e^{2\rho(\log a)} = \prod_{i=1}^{r} t_i^{\frac{d}{2}(r-2i+1)}.$$

Since $\mathcal{D} = G/K$, we can normalize the Haar measure of G by the relation $dg = d_{\mathcal{D}}(u)\cdot dk$ where $u = ge$. Then by (2.15), (2.16), (2.17) one has

$$\begin{aligned} \Gamma_{\mathcal{D}}(s) &= \int_G N(ge)^s\, e^{-\tau(ge)}\, dg \\ &= c_1 \int_A \det(a)^s\, e^{-2\rho(\log a)}\, da \int_N e^{-\tau(\underline{n}\cdot ae)}\, d\underline{n} \\ &= c_1 \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} (t_i^{\frac{n}{r}s-\frac{d}{2}(r-2i+1)-1}\, dt_i) \\ &\qquad \times \int_N e^{-\tau(\underline{n}(\sum t_i e_i))}\, d\underline{n} \end{aligned} \tag{2.18}$$

To compute the integral over N, we introduce some notations. For $u = \sum_{i<j} u_{ij} \in U$ with $u_{ij} \in U_{ij}$, we put

$$\begin{aligned} T_u^{(+)} &= \tfrac{1}{2}(T_u + \sum_{i<j} [T_{e_i-e_j}, T_{u_{ij}}]), \\ \varepsilon^{(+)}(u) &= \sum_{i<j} \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \sum_{i<k_1<\cdots<k_{\nu-1}<j} u_{ik_1} u_{k_1k_2} \cdots u_{k_{\nu-1}j}. \end{aligned} \tag{2.19}$$

Then one has The U_{ij}-component of $\varepsilon^{(+)}(u)$ is denoted by $\varepsilon_{ij}^{(+)}(u)$.

LEMMA 2. The notation being as above, one has

$$(\exp T_u^{(+)})(\sum_{i=1}^{r} t_i e_i) = \sum_{i=1}^{r} (t_i + \tfrac{1}{4} \sum_{k>i} t_k\, \varepsilon_{ik}^{(+)}(u)^2) e_i + \tfrac{1}{2} \sum_{i<j} (t_j\, \varepsilon_{ij}^{(+)}(u) + \sum_{k>j} t_k\, \varepsilon_{ik}^{(+)}(u)\, \varepsilon_{jk}^{(+)}(u)) \tag{2.20}$$

This may be regarded as a generalization of the so-called "Jacobi transformation". The proof is again straightforward. It follows that, if $\underline{n} = \exp T_u^{(+)}$ $(u \in \sum_{i<j} U_{ij})$, one has

$$\tau(\underline{n}(\sum_i t_i e_i)) = \frac{n}{r} \sum_i t_i + \frac{1}{8} \sum_{i<k} \tau(\varepsilon_{ik}^{(+)}(u)^2) t_k. \tag{2.21}$$

We denote the Euclidean measure on U_{ij} $(i<j)$ (relative to the inner product $\langle\ \rangle$) by du_{ij} and define the Haar measure on N by

$$d\underline{n} = \prod_{i<j} du_{ij} \qquad \text{for } \underline{n} = \exp T_u^{(+)}.$$

Since the map $\varepsilon^{(+)}$ is a bijection of $\sum_{i<j} U_{ij}$ onto itself with jacobian

equal to one, one has

$$du = \prod_{i<j} du_{ij} = \prod_{i<j} du'_{ij},$$

where $u' = \varepsilon^{(+)}(u)$. Hence by (2.21) one has

$$\begin{aligned}\int_N e^{-\tau(\underline{n} \sum t_i e_i)}\, d\underline{n} &= e^{-\frac{n}{r}\sum t_i} \prod_{i<j} \int_{U_{ij}} e^{-\frac{t_j}{8}\tau(u'^2_{ij})} du'_{ij} \\ &= e^{-\frac{n}{r}\sum t_i} \prod_{i<j} \left(\frac{8\pi}{t_j}\right)^{\frac{d}{2}} \\ &= (8\pi)^{\frac{n-r}{2}} \prod_j \left(t_j^{-\frac{d}{2}(j-1)} e^{-\frac{n}{r}t_j}\right).\end{aligned}$$

Inserting this in (2.18), one obtains

$$\begin{aligned}\Gamma_{\Omega}(s) &= c_1 (8\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \left(\int_0^\infty t_j^{\frac{n}{r}s - \frac{d}{2}(r-j)-1} e^{-\frac{n}{r}t_j}\, dt_j\right) \\ &= c_1 (8\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \left(\frac{n}{r}\right)^{-\frac{n}{r}s + \frac{d}{2}(r-j)} \Gamma\left(\frac{n}{r}s - \frac{d}{2}(r-j)\right) \\ &= c_1 (8\pi)^{\frac{n-r}{2}} \left(\frac{n}{r}\right)^{-ns+\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\left(\frac{n}{r}s - \frac{d}{2}(j-1)\right).\end{aligned}$$

The constant c_1 can be determined by the following observation. We set

$$U_0 = \sum_{i=1}^{r} U_{ii} = \{e_1, \ldots, e_r\}_R$$

and denote by du_0 the Euclidean measure on U_0 (relative to $\langle\ \rangle$). Then, since $\langle e_i, e_j\rangle = \frac{n}{r}\delta_{ij}$, the bijection $A \longrightarrow U_0$ defined by $a = \exp T_{u_0}$, or equivalently by $ae = \exp u_0$, gives the relation

$$du_0 = \left(\frac{n}{r}\right)^{\frac{r}{2}} da.$$

Hence, when

$$u = (\underline{n}a)e = \underline{n}\left(\sum_i t_i e_i\right),$$

$$\underline{n} = \exp T_x^{(+)}, \quad x \in \sum_{i<j} U_{ij}, \quad x' = \varepsilon^{(+)}(x),$$

one has by Lemma 2

$$\frac{\partial(u)}{\partial(t, x)} = \frac{\partial(u_0, u_{ij})}{\partial(t_i, x'_{ij})} = \left(\frac{n}{r}\right)^{\frac{r}{2}} \prod_{j=1}^{r} \left(\frac{t_j}{2}\right)^{(j-1)d}$$

$$= 2^{r-n} \left(\frac{n}{r}\right)^{\frac{r}{2}} \prod_{j=1}^{r} t_j^{(j-1)d}.$$

It follows that

$$d_{\mathcal{D}}(u) = 2^{r-n}\left(\frac{n}{r}\right)^{\frac{r}{2}} \prod_{j=1}^{r} (t_j^{(j-1)d-\frac{n}{r}} dt_j)\, dx,$$

which, in view of (2.11) and (2.16), implies (2.15) and the relation

$$c_1 = 2^{r-n} \left(\frac{n}{r}\right)^{\frac{r}{2}}. \tag{2.22}$$

Thus we obtain the formula

$$\Gamma_{\mathcal{D}}(s) = (2\pi)^{\frac{n-r}{2}} \left(\frac{n}{r}\right)^{n(\frac{1}{2}-s)} \prod_{j=1}^{r} \Gamma\left(\frac{n}{r} s - \frac{d}{2}(j-1)\right). \tag{2.23}$$

Our computation also shows that the integral for $\Gamma_{\mathcal{D}}(s)$ converges absolutely for $\operatorname{Re} s > 1 - \frac{r}{n}$.

From the relation (1.2) one obtains

$$\Gamma_{\mathcal{D}}(s)\, \Gamma_{\mathcal{D}}(1-s) = (2\pi)^{n-r} \prod_{j=1}^{r} \Gamma\left(\frac{n}{r} s - \frac{d}{2}(j-1)\right) \Gamma\left(\frac{n}{r}(1-s) - \frac{d}{2}(r-j)\right)$$

$$= (2\pi)^{n-r}(2\pi i)^{r} \prod_{j=1}^{r} \frac{e^{\pi i(\frac{n}{r} s-\frac{d}{2}(j-1))}}{e^{2\pi i(\frac{n}{r} s-\frac{d}{2}(j-1))} - 1}$$

Since one has by (2.11)

$$n - r = d\,\frac{r(r-1)}{2} \equiv \begin{cases} 0 \pmod 2 & \text{for } d \text{ even} \\ \left[\frac{r}{2}\right] \pmod 2 & \text{for } d \text{ odd,} \end{cases}$$

one has

$$\prod_{j=1}^{r} e^{-\pi i \frac{d}{2}(j-1)} = (-i)^{d\frac{r(r-1)}{2}} = \begin{cases} i^{n-r} & \text{for } d \text{ even} \\ i^{n-r}(-1)^{[\frac{r}{2}]} & \text{for } d \text{ odd.} \end{cases}$$

Hence one obtains the following functional equation:

$$\Gamma_{\mathcal{D}}(s)\, \Gamma_{\mathcal{D}}(1-s) = (2\pi i)^{n} e^{n\pi i s} \begin{cases} (e^{2\pi i \frac{n}{r} s} - 1)^{-r} & (d \text{ even}) \\ (e^{2\pi i \frac{n}{r} s} - 1)^{-[\frac{r+1}{2}]}(e^{2\pi i\frac{n}{r} s}+1)^{-[\frac{r}{2}]} & (d \text{ odd}). \end{cases} \tag{2.24}$$

§ 3. Zeta functions of a self-dual homogeneous cone.

3.1. We fix a $\mathbb{Q}$-structure on U and assume that (the Zariski closure of) G is defined over $\mathbb{Q}$ and $e \in U_{\mathbb{Q}}$; then (the Zariski closure of) K is also defined over $\mathbb{Q}$. We also fix a lattice L in U compatible with that $\mathbb{Q}$-structure, i.e., such that $U_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$, and an arithmetic subgroup Γ fixing L, i.e., a subgroup of $G_L = \{ g \in G \mid gL = L \}$ of finite index; for simplicity we assume that Γ has no fixed point in Ω. We then define the zeta function associated with Ω, Γ, L as follows:

$$\zeta_{\Omega}(s; \Gamma, L) = \sum_{u \in \Gamma \backslash \Omega \cap L} N(u)^{-s}, \tag{3.1}$$

the summation being taken over a complete set of representatives of $\Omega \cap L$ modulo Γ. It can be shown easily that this series is absolutely convergent for $\operatorname{Re} s > 1$.

By the reduction theory, Γ has a fundamental domain in Ω which is a rational polyhedral cone. More precisely, there exists a finite set of simplicial cones

$$\begin{aligned} C^{(i)} &= \{ v_1^{(i)}, \dots, v_{\ell_i}^{(i)} \}_{\mathbb{R}_+} \\ &= \Big\{ \sum_{j=1}^{\ell_i} \lambda_j v_j^{(i)} \;\Big|\; \lambda_j \in \mathbb{R}_+ \Big\} \qquad (1 \leq i \leq m), \end{aligned}$$

where $v_1^{(i)}, \dots, v_{\ell_i}^{(i)}$ are linearly independent elements in $\bar{\Omega} \cap L$, such that

$$\Omega = \coprod_{\substack{\gamma \in \Gamma \\ 1 \leq i \leq m}} \gamma C^{(i)}.$$

It follows that

$$\zeta(s; \Gamma, L) = \sum_{i=1}^{m} \sum_{u \in C^{(i)} \cap L} N(u)^{-s}.$$

For a set of linearly independent vectors $v_1, \dots, v_\ell \in L$, we put

$$R((v_j), L) = \Big\{ \sum_{j=1}^{\ell} \lambda_j v_j \;\Big|\; 0 < \lambda_j \leq 1 \Big\} \cap L,$$

which is finite. Then $u \in C^{(i)} \cap L$ can be written uniquely in the form

$$u = v_0 + \sum_{j=1}^{\ell_i} m_j v_j^{(i)}, \quad v_0 \in R((v_j^{(i)}), L),\ m \in \mathbb{Z},\ m \geq 0.$$

For a set of linearly independent vectors $v_1, \ldots, v_\ell \in \mathfrak{D} \cap V_Q$ and $v_0 = \sum_j \alpha_j v_j \in \mathfrak{D}$ $(\alpha_j \in \mathbb{Q}_+)$, we define a "partial zeta function" by

$$(3.2) \qquad \zeta_{\mathfrak{D}}(s;\,(v_j),\, v_0) = \sum_{m_j \geq 0} N(v_0 + \sum_{j=1}^{\ell} m_j v_j)^{-s},$$

which will also be written as $\zeta_{\mathfrak{D}}(s;\,(v_j),\,(\alpha_j))$. Then the zeta function (3.1) can be written as a finite sum of partial zeta functions as follows:

$$(3.3) \qquad \zeta_{\mathfrak{D}}(s;\,\Gamma,\, L) = \sum_{i=1}^{m} \sum_{v_0 \in R((v_j^{(i)}), L)} \zeta_{\mathfrak{D}}(s;\,(v_j^{(i)}),\, v_0).$$

Hence the study of special values of $\zeta_{\mathfrak{D}}(s;\,\Gamma,\,L)$ is reduced to that of the partial zeta functions of the form (3.2).

3.2. Let (v_j) and v_0 be as above. Then by (2.7) one obtains

$$\begin{aligned}
\Gamma_{\mathfrak{D}}(s)\zeta_{\mathfrak{D}}(s;\,(v_j),\, v_0) &= \sum_{\substack{m_j \geq 0 \\ 1 \leq j \leq \ell}} \Gamma_{\mathfrak{D}}(s) N(v_0 + \sum_{j=1}^{\ell} m_j v_j)^{-s} \\
&= \sum_{\substack{m_j \geq 0 \\ 1 \leq j \leq \ell}} \int_{\mathfrak{D}} N(u)^{s-1} e^{-\sum_j (\alpha_j + m_j)\langle v_j,\, u\rangle} du \\
&= \int_{\mathfrak{D}} N(u)^{s} \prod_{j=1}^{\ell} b(\langle v_j,\, u\rangle,\, 1-\alpha_j)\, d_{\mathfrak{D}}(u) \\
&= \int_{G} \det(g)^{s} \prod_{j=1}^{\ell} b(\langle v_j,\, ge\rangle,\, 1-\alpha_j)\, dg.
\end{aligned}$$

In the notation of § 2, but this time using the decomposition $G = KAK$, one has

$$(3.4) \qquad dg = c\, \Delta(a)\, dk\cdot da\cdot dk'$$

for $g = kak'$, $k, k' \in K$, $a \in A$. Here c is a positive constant and

$$\begin{aligned}
\Delta(a) &= \prod_{\alpha \in \Phi_+} (e^{\alpha(\log a)} - e^{-\alpha(\log a)})^{d} \\
&= (\prod_{i=1}^{r} t_i)^{-\frac{d}{2}(r-1)} \,|\Delta(t_1, \ldots, t_r)|^{d},
\end{aligned}$$

where $\Delta(t_1, \ldots, t_r) = \prod_{i<j} (t_i - t_j)$ (cf. Helgason [4], Ch. X, § 1). Hence in view of (2.11) and (2.17) one has

$$(3.5) \qquad \Gamma_{\mathfrak{D}}(s)\, \zeta_{\mathfrak{D}}(s;\,(v_j),\,(\alpha_j)) = c \int_0^{\infty} \cdots \int_0^{\infty} (\prod_{i=1}^{r} t_i)^{\frac{n}{r}(s-1)} |\Delta(t)|^{d} F(t) \prod_{i=1}^{r} dt_i,$$

where

$$F(t_1, \ldots, t_r) = \int_K \prod_{j=1}^{\ell} b(\langle v_j, k \sum t_i e_i \rangle, 1 - \alpha_j)\, dk.$$

It is clear that $F(t_1, \ldots, t_r)$ is holomorphic for $\mathrm{Re}\, t_i > 0$ $(1 \leq i \leq r)$.

Since K contains an element which induces any given permutation of $e_1, \ldots, e_r$, the function F is symmetric. Hence, denoting by B_1 an open simplicial cone in R^r defined by $t_1 > \ldots > t_r > 0$, one has

$$(3.5') \qquad \Gamma_{\Omega}(s)\, \zeta_{\mathcal{L}}(s; (v_j), (\alpha_j)) = c\, r! \int_{B_1} (\prod t_i)^{\frac{n}{r}(s-1)} \Delta(t)^d F(t) \prod dt_i .$$

3.3. Still following Shintani [11], we make a change of variables $(t_i) \longrightarrow (t_1, \tau_2, \ldots, \tau_r)$ with $\tau_i = t_i/t_{i-1}$ $(2 \leq i \leq r)$. Then B_1 can be expressed as

$$B_1 = \left\{ (t_i) \;\middle|\; t_i = t_1 \prod_{j=2}^{i} \tau_j , \; 0 < t_1 < \infty, \; 0 < \tau_i < 1 \right\}.$$

Putting $\tau_1 = t_1$, one has

$$\frac{\partial(t_1, \ldots, t_r)}{\partial(t_1, \tau_2, \ldots, \tau_r)} = \prod_{i=1}^{r} \tau_i^{r-i},$$

$$\prod t_i = \prod \tau_i^{r-i+1},$$

$$\Delta(t) = \prod \tau_i^{\frac{1}{2}(r-i+1)(r-i)} \prod_{2 \leq i < j \leq r} (1 - \tau_i \cdots \tau_j).$$

It follows that the exponent of τ_i in the integrand in (3.5') is equal to

$$(r-i+1)\tfrac{n}{r}(s-1) + \tfrac{d}{2}(r-i+1)(r-i) + r - i$$

$$= (r-i+1)\left\{ \tfrac{n}{r} s - \tfrac{d}{2}(i-1) \right\} - 1.$$

Hence one has

$$(3.6) \qquad \Gamma_{\mathcal{L}}(s)\, \zeta_{\mathcal{L}}(s; (v_j), (\alpha_j)) = c\, r! \int_0^{\infty} t^{ns-1}\, dt$$
$$\int_0^1 \cdots \int_0^1 \prod \tau_i^{(r-i+1)\{\frac{n}{r}s - \frac{d}{2}(i-1)\} - 1} \tilde{F}(t_1, \tau) \prod_{i=2}^{r} d\tau_i,$$

where

$$(3.7) \qquad \tilde{F}(t_1, \tau) = \prod_{2 \leq i < j \leq r} (1 - \tau_i \cdots \tau_j)^d F(t_1, t_1\tau_2, \ldots, t_1\tau_2 \cdots \tau_r).$$

3.4. We now assume that all v_j's are in $\mathcal{D}$ (not on the boundary of $\mathcal{D}$). (In the situation explained in 3.1, this means that the $\mathbb{Q}$-rank of G is equal to 1.) Then for any $v \in \bar{\mathcal{D}} - \{0\}$, one has $< v_j , v > \, > 0$; in particular,

(3.8) $$< v_j , ke_i > \, > 0 \quad \text{for all } k \in K,\ 1 \leq i \leq r.$$

Put

(3.9)
$$\begin{aligned} \xi_j &= < v_j , k \sum t_i e_i > \\ &= t_1 < v_j , k(e_1 + \sum_{i=2}^{r} \tau_2 \cdots \tau_i e_i) > \\ &= t_1 < v_j , ke_1 > (1 + \sum_{i=2}^{r} \tau_2 \cdots \tau_i \frac{< v_j , ke_i >}{< v_j , ke_1 >}). \end{aligned}$$

For the fixed e_i, v_j, choose ρ, $\rho_1 > 0$ in such a way that

(3.10)
$$\begin{cases} \displaystyle\sum_{i=2}^{r} \rho^{i-1} \frac{< v_j , ke_i >}{< v_j , ke_1 >} < 1 & \text{for all } k \in K,\ 1 \leq j \leq \ell, \\ \rho_1 < \dfrac{\pi}{< v_j , ke_1 >} & \text{for all } 1 \leq j \leq \ell . \end{cases}$$

Then for

(3.11) $$0 < |t_1| < \rho_1 , \quad |\tau_i| < \rho \qquad (2 \leq i \leq r),$$

one has $0 < |\xi_j| < 2\pi$ and so $b(\xi_j , 1 - \alpha_j)$ is holomorphic. Hence the function $F(t) = F(t_1 , t_1\tau_2 , \ldots, t_1\tau_2 \cdots \tau_r)$ has a Laurent expansion in $t_1, \tau_2 , \ldots, \tau_r$ in the domain defined by (3.11). The coefficients in this expansion is a Q-linear combination of the integrals of the form

(3.12) $$I((\nu_{ij})) = \int_K \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq \ell}} < v_j , ke_i >^{\nu_{ij}} dk$$

where $\nu_{ij} \geqslant 0$ for $2 \leq i \leq r$ and $\nu_{ij} \in \mathbb{Z}$ for all i, j.

3.5. Let $I(\varepsilon, 1)$ denote the contour consisting of the line segment $[\varepsilon, 1]$ taken twice in opposite directions and of a (small) circle of radius ε about the origin taken in the counterclockwise direction. When the τ_i $(2 \leq i \leq r)$ are on $I(\varepsilon, 1)$, one has by (2.12)

$$\left| < v_j , k(e_1 + \sum_{i=2}^{r} \tau_2 \cdots \tau_i e_i) > \right| \leq |v_j| \cdot \sum_{i=1}^{r} |e_i| = \sqrt{nr}\, |v_j|$$

and

$$\begin{aligned} \mathrm{Re} < v_j , k(e_1 + \sum_{i=2}^{r} \tau_2 \cdots \tau_i e_i) > &= < v_j , ke_1 > + \sum_{i=2}^{r} \mathrm{Re}(\tau_2 \cdots \tau_i) < v_j , ke_i > \\ &\geq < v_j , ke_1 > - \varepsilon |v_j| \sum_{i=2}^{r} |e_i| \\ &= < v_j , ke_1 > - \varepsilon (r-1) \sqrt{\tfrac{n}{r}}\, |v_j| . \end{aligned}$$

We choose ε so that one has

$$(3.13) \qquad \varepsilon \sqrt{rn}\, |v_j| < \mathrm{Min} \left\{ 2\pi , < v_j , ke_1 > \ (k \in K) \right\} \qquad \text{for all } 1 \leq j \leq \ell ,$$

Then the above inequalities show that $< v_j , k(e_1 + \sum_{i=2}^{r} \tau_2 \cdots \tau_i e_i) >$ belongs to the domain

$$\left\{ z \in \mathbf{C} \;\middle|\; |z| < \frac{2\pi}{\varepsilon} , \ \mathrm{Re}\, z > \varepsilon \sqrt{\tfrac{n}{r}}\, |v_j| \right\} .$$

It follows that, if t_1 is on the contour $I(\varepsilon, \infty)$, one has

$$|\xi_j| < 2\pi \qquad \text{or} \qquad \mathrm{Re}\, \xi_j > 0 ,$$

so that the function $b(\xi_j , 1 - \alpha_j)$ is holomorphic.

From this observation, it is clear that the integral on the r.h.s. of (3.6) is equal to the contour integral

$$(e^{2\pi i n s} - 1)^{-1} \int_{t_1 \in I(\varepsilon, \infty)} \cdot \prod_{i=2}^{r} (e^{2\pi i \frac{r-i+1}{r} ns} - 1)^{-1} \int_{\tau_i \in I(\varepsilon, 1)}$$

which is independent of the choice of ε satisfying (3.13). As is easily seen, the contour integral converges for all $s \in \mathbf{C}$. Hence the above integral ~~[illegible]~~, viewed as a function in s, can be continued to a meromorphic function on the whole plane; the possible poles are of the form $\dfrac{r\nu}{(r-i+1)n}$ $(\nu \in \mathbf{Z})$.

§ 4. The special values of the zeta functions.

4.1. As a preliminary, we check the rationality of the constant c in (3.4). For that purpose, we compute $\Gamma_{\mathcal{D}}(s)$ by using the decomposition $G = KAK$.

$$
\begin{aligned}
(4.1)\qquad \Gamma_{\mathcal{D}}(s) &= \int_{\mathcal{D}} N(u)^s \, e^{-\tau(u)} \, d_{\mathcal{D}}(u) \\
&= \int_{G} N(ge)^s \, e^{-\tau(ge)} \, dg \\
&= c \int_{A} \det(a)^s \, e^{-\tau(ae)} \, \Delta(a) \, da \\
&= c \int_0^\infty \cdots \int_0^\infty \left(\prod t_i \right)^{\frac{n}{r}(s-1)} |\Delta(t)|^d \, e^{-\frac{n}{r}\sum t_i} \prod dt_i .
\end{aligned}
$$

We make another change of variables:

$$t = \sum_{i=1}^{r} t_i , \quad t_i' = t_i/t.$$

Then

$$\frac{\partial(t_1, \ldots, t_r)}{\partial(t, t_1', \ldots, t_{r-1}')} = (-1)^{r-1} \, t^{r-1},$$

and the exponent of t in the integrand in the last member of (4.1) is equal to

$$n(s - 1) + \frac{d}{2} r(r - 1) + r - 1 = ns - 1.$$

Hence one has

$$(4.2)\qquad \Gamma_{\mathcal{D}}(s) = c \cdot \gamma(s) \cdot \beta(s),$$

where

$$
(4.3)\qquad \left\{
\begin{aligned}
\gamma(s) &= \int_0^\infty t^{ns-1} \, e^{-\frac{n}{r}t} \, dt = \left(\frac{r}{n}\right)^{ns} \Gamma(ns), \\
\beta(s) &= \int_{\substack{t_i' > 0 \\ \sum t_i' < 1}} \left\{ t_1' \cdots t_{r-1}' \left(1 - \sum t_i'\right) \right\}^{\frac{n}{r}(s-1)} \times \\
&\qquad |\Delta(t_1', \ldots, t_{r-1}', 1 - \sum t_i')|^d \prod dt_i' .
\end{aligned}
\right.
$$

For $s = 1$, one has

$$\Gamma_{\mathcal{D}}(1) = c \cdot \gamma(1) \cdot \beta(1) = c \left(\frac{r}{n}\right)^n (n-1)! \, \beta(1),$$

$$\beta(1) = \int_{\substack{t_i' > 0 \\ \sum t_i' < 1}} |\Delta(t_1', \ldots, t_{r-1}', 1-\sum t_i')|^d \prod dt_i' \in \mathbb{Q}.$$

By (2.23) one has

(4.4) $$\Gamma_{\mathcal{D}}(1) = (2\pi)^{\frac{n-r}{2}} (\tfrac{r}{n})^{\frac{n}{2}} \prod_{j=1}^{r} \Gamma(1 + \tfrac{d}{2}(j-1))$$

$$\underset{\mathbb{Q}}{\sim} \begin{cases} \pi^{\frac{n-r}{2}} & (d \text{ even}) \\ \pi^{\frac{1}{2}(n-[\frac{r+1}{2}])} & (d \text{ odd}), \end{cases}$$

where $a \underset{\mathbb{Q}}{\sim} b$ means that $a/b \in \mathbb{Q}$. Thus one has

(4.5) $$c = \frac{(2\pi)^{\frac{n-r}{2}} (\frac{n}{r})^{\frac{n}{2}} \prod_{j=1}^{r} \Gamma(1 + \frac{d}{2}(j-1))}{(n-1)!\ \beta(1)} \underset{\mathbb{Q}}{\sim} \Gamma_{\mathcal{D}}(1).$$

Since $\Gamma_{\mathcal{D}}(1) \underset{\mathbb{Q}}{\sim} \Gamma_{\mathcal{D}}(1 + \frac{r}{n}\nu)$ for $\nu \in \mathbb{Z}$, one obtains

(4.6) $$c\,\Gamma_{\mathcal{D}}(1 + \tfrac{r}{n}\nu) \underset{\mathbb{Q}}{\sim} \Gamma_{\mathcal{D}}(1)^2 \underset{\mathbb{Q}}{\sim} \begin{cases} \pi^{n-r} & (d \text{ even}) \\ \pi^{n-[\frac{r+1}{2}]} & (d \text{ odd}). \end{cases}$$

4.2. We first consider the case where d is even. Then by (2.24) one has

$$\Gamma_{\mathcal{D}}(s)\,\Gamma_{\mathcal{D}}(1-s) = (2\pi i)^n e^{\pi i n s}(e^{2\pi i \frac{n}{r} s} - 1)^{-r}.$$

Hence

(4.7) $$\zeta_{\mathcal{D}}(s; (v_j), (\alpha_j)) = \frac{c\,\Gamma_{\mathcal{D}}(1-s)}{(2\pi i)^{n-r} e^{\pi i n s}} \times R(s),$$

where

$$R(s) = \left(\frac{e^{2\pi i \frac{n}{r} s} - 1}{2\pi i}\right)^r \cdot r! \int_{B_1} (\prod t_i)^{\frac{n}{r}(s-1)} \Delta(t)^d F(t) \prod dt_i$$

$$= \prod_{j=1}^{r} \frac{e^{2\pi i \frac{n}{r} s} - 1}{e^{2\pi i \frac{r-j+1}{r} ns} - 1} \times \frac{1}{(2\pi i)^r} \int_{I(\varepsilon,\infty)} t_1^{ns-1}\, dt_1$$

$$\Big(\prod_{r=2}^{r} \int_{\tau_i \in I(\varepsilon,1)}\Big) \prod_{i=2}^{r} \tau_i^{(r-i+1)\{\frac{n}{r} s - \frac{d}{2}(i-1)\} - 1}\, r!\, \tilde{F}(t_1, \tau_i) \prod d\tau_i.$$

We are interested in the values of $\zeta_{\mathcal{D}}$ at $s = -\frac{r}{n}\nu$ ($\nu = 0, 1, \ldots$). The first factor in the right hand side of (4.7) is holomorphic for $\operatorname{Re} s < \frac{r}{n}$ and by (4.6) the value at $s = -\frac{r}{n}\nu$ is rational:

$$(4.9) \qquad \frac{c\,\Gamma_{\mathcal{D}}(1+\frac{r}{n}\nu)}{(2\pi i)^{n-r}\, e^{-r\nu\pi i}} = (-1)^{\frac{n-r}{2}+r\nu}\,\frac{c\,\Gamma_{\mathcal{D}}(1+\frac{r}{n}\nu)}{(2\pi)^{n-r}} \in Q.$$

On the other hand, it is clear that

$$\frac{e^{2\pi i\frac{n}{r}s}-1}{e^{2\pi i\frac{r-i+1}{r}ns}-1} \longrightarrow \frac{1}{r-i+1} \qquad \text{when } s \longrightarrow -\frac{r}{n}\nu .$$

Hence we see that $R(-\frac{r}{n}\nu)$ is equal to the coefficient of

$$t_1^{r\nu}\prod_{i=2}^{r}\tau_i^{(r-i+1)\{\nu+\frac{d}{2}(i-1)\}}$$

in the Laurent expansion of $\widetilde{F}(t_1,\tau)$,

which is a Q-linear combination of $I((\nu_{ij}))$.

4.3. From now on we assume that d is odd. By the classification theory, it is known that this assumption implies that $r = 2$ $(n = d+2)$ or $d = 1$ $(n = \frac{1}{2}r(r+1))$. By (2.24) one has

$$\Gamma_{\mathcal{D}}(s)\,\Gamma_{\mathcal{D}}(1-s) = (2\pi i)^n\, e^{n\pi i s}\,(e^{2\pi i\frac{n}{r}s}-1)^{-[\frac{r+1}{2}]}(e^{2\pi i\frac{n}{r}s}+1)^{-[\frac{r}{2}]}.$$

Hence

$$(4.11) \qquad \zeta_{\mathcal{D}}(s;(v_j),(\alpha_j)) = \frac{c\,\Gamma_{\mathcal{D}}(1-s)}{(2\pi i)^{n-[\frac{r+1}{2}]}e^{n\pi i s}} \times R^{(1)}(s)\,R^{(2)}(s),$$

where

$$R^{(1)}(s) = (2\pi i)^{[\frac{r}{2}]}\, r!\,\frac{(e^{2\pi i\frac{n}{r}s}-1)^{[\frac{r+1}{2}]}(e^{2\pi i\frac{n}{r}s}+1)^{[\frac{r}{2}]}}{\prod_{k=1}^{r}(e^{2\pi i(r-k+1)\{\frac{n}{r}s-\frac{d}{2}(k-1)\}}-1)},$$

$$R^{(2)}(s) = (2\pi i)^{-r}\int_{I(\varepsilon,\infty)} t_1^{ns-1}dt_1 \int\cdots\int_{I(\varepsilon,1)^{r-1}} \prod \tau_i^{(r-i+1)\{\frac{n}{r}s-\frac{d}{2}(i-1)\}-1}\widetilde{F}(t_1,\tau)\prod d\tau_i .$$

The first factor in the right hand side of (4.11) is holomorphic for Re s $< \frac{r}{n}$ and by (4.6) the value at $s = -\frac{r}{n}\nu$ $(\nu \geqslant 0)$ is rational:

$$(4.12) \qquad \frac{c\,\Gamma_{\mathcal{D}}(1+\frac{r}{n}\nu)}{(2\pi i)^{n-[\frac{r+1}{2}]}\,e^{-\pi i r\nu}} = (-1)^{\frac{1}{2}(n-[\frac{r+1}{2}])+r\nu}\,\frac{c\,\Gamma_{\mathcal{D}}(1+\frac{r}{n}\nu)}{(2\pi)^{n-[\frac{r+1}{2}]}} \in \mathbb{Q}.$$

Note that one has

$$n \equiv [\tfrac{r+1}{2}] \qquad (\mathrm{mod}\ 2),$$

since

$$n = d+2 \equiv 1 = [\tfrac{3}{2}] \quad (\mathrm{mod}\ 2) \qquad \text{if } r = 2, \text{ and}$$

$$n = \tfrac{1}{2}r(r+1) \equiv [\tfrac{r+1}{2}] \quad (\mathrm{mod}\ 2) \quad \text{if } d = 1.$$

4.4. To compute $R^{(1)}(s)$, we first note

$$e^{\pi i d(k-1)(r-k+1)} = \begin{cases} -1 & \text{if } k \equiv r \equiv 0 \ (\mathrm{mod}\ 2), \\ 1 & \text{otherwise.} \end{cases}$$

We put

$$[\tfrac{r}{2}] = r_1, \qquad \zeta = e^{2\pi i \frac{n}{r} s}.$$

<u>The case</u> r <u>is odd</u>. One has

$$R^{(1)}(s) = (2\pi i)^{r_1}\, r!\, \frac{(\zeta-1)^{r_1+1}(\zeta+1)^{r_1}}{\prod_{k=1}^{r}(\zeta^k - 1)}$$

$$= \frac{r!}{\prod_{k=1}^{r}(\zeta^{k-1}+\dots+\zeta+1)}\,\left(2\pi i\cdot\frac{\zeta+1}{\zeta-1}\right)^{r_1}.$$

Hence, when $s \longrightarrow -\frac{r}{n}\nu$, one has

$$(4.13) \qquad (s+\tfrac{r}{n}\nu)^{r_1} R^{(1)}(s) \longrightarrow (2\tfrac{r}{n})^{r_1}.$$

Thus $R^{(1)}(s)$ has a pole of order r_1 at $s = -\frac{r}{n}\nu$.

<u>The case</u> r <u>is even</u>. One has

$$R^{(1)}(s) = (2\pi i)^{r_1}\, r!\, \frac{(\zeta-1)^{r_1}(\zeta+1)^{r_1}}{\prod_{k=1}^{r}\{(-1)^k\zeta^k - 1\}}$$

$$= (-2\pi i)^{r_1} \frac{r!}{\prod_{\substack{1\le k\le r\\ k \text{ even}}} (\zeta^{k-1}+\ldots+\zeta+1) \prod_{\substack{1\le k\le r\\ k \text{ odd}}} (\zeta^{k-1}-\ldots-\zeta+1)}$$

Hence $R^{(1)}$ is holomorphic at $s = -\frac{r}{n}\nu$ and

(4.14) $R^{(1)}(-\frac{r}{n}\nu) = (-2\pi i)^{r_1} \dfrac{r!}{(2r_1)!!} = (-\pi i)^{r_1} \dfrac{r!}{r_1!}$.

4.5. When r is odd (hence $d = 1$, $n = \frac{1}{2}r(r+1)$), $R^{(2)}(s)$ for $s = -\frac{r}{n}\nu = -\frac{2\nu}{r+1}$ is given by the coefficient of

$$t_1^{r\nu} \prod_{i=2}^{r} \tau_i^{(r-i+1)(\nu+\frac{i-1}{2})}$$

in the Laurent expansion of $\tilde{F}(t_1, \tau)$. Hence $\zeta_{\mathcal{L}}(s; (v_j), (\alpha_j))$ has at most a pole of order $r_1 = \frac{r-1}{2}$ at $s = -\frac{2\nu}{r+1}$ and one has

(4.15) $\lim_{s\to -\frac{2\nu}{r+1}} (s + \frac{2\nu}{r+1})^{r_1} \zeta_{\mathcal{L}}(s; (v_j), (\alpha_j)) \underset{Q}{\sim} R^{(2)}(-\frac{2\nu}{r+1})$.

To treat the case r is even, we use the formula

$$\int_{I(\varepsilon,1)} t^{\frac{m}{2}-1}\, dt = -\frac{4}{m} \qquad (m \text{ odd}),$$

which can be verified easily. When r is even, the value of $R^{(2)}(s)$ for $s = -\frac{r}{n}\nu$ is given by

(4.16) $$(-\pi i)^{-r_1} \sum_{m_1,\ldots,m_{r_1}\in\mathbb{Z}} \frac{a_{(m_j)}}{\prod_{j=1}^{r_1}(m_j - (r-2j+1)\{\nu + \frac{d}{2}(2j-1)\})}$$

where $a_{(m_j)}$ is the coefficient of

$$t_1^{r\nu} \prod_{j=2}^{r_1} \tau_{2j-1}^{(r-2j+2)(\nu+d(j-1))} \prod_{j=1}^{r_1} \tau_{2j}^{m_j}$$

in $\tilde{F}(t_1, \tau)$. Hence for the value of $\zeta_{\mathcal{L}}$, one has

(4.17) $\zeta_{\mathcal{L}}(-\frac{r}{n}\nu; (v_j), (\alpha_j)) \underset{Q}{\sim} (2\pi i)^{r_1} R^{(2)}(-\frac{r}{n}\nu)$.

Bibliography

[1] A. Ash et al., Smooth Compactification of Locally Symmetric Varieties, Math. Sci. Press, Brookline, 1975

[2] H. Braun and M. Koecher, Jordan-Algebren, Springer-Verlag, 1966.

[3] S. G. Gindikin, Analysis in homogeneous domains, Uspehi Mat. Nauk 19 (1964), 3-92; = Russian Math. Survey 19 (1964), 1-89.

[4] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Acad. Press, 1978.

[5] M. Koecher, Positivitätsbereiche im $\mathbb{R}^n$, Amer. J. Math. 79 (1957), 575-596.

[6] H.L. Resnikoff, On a class of linear differential equations for automorphic forms in several complex variables, Amer. J. Math. 95 (1973), 321-331.

[7] I. Satake, Algebraic Structures of Symmetric Domains, Iwanami-Shoten and Princeton Univ. Press, 1980.

[8] M. Sato and T. Shintani, On zeta functions associated with prehomogeneous vector spaces, Ann. of Math. 100 (1974), 131-170.

[9] T. Shintani, On Dirichlet series whose coefficients are class-numbers of integral binary cubic forms, J. Math. Soc. Japan 24 (1972), 132-188.

[10] -----, On zeta-functions associated with the vector space of quadratic forms, J. Fac. Sci. Univ. Tokyo 22 (1975), 25-65.

[11] -----, On evaluation of zeta functions of totally real algebraic number fields at non-positive integers, J. Fac. Sci. Univ. Tokyo 23 (1976), 393-417.

[12] C. L. Siegel, Berechnung von Zetafunktionen an ganzzahligen Stellen, Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. 1968, 7-38.

[13] -----, Über die Zetafunktionen indefiniter quadratischer Formen, Math. Z. 43 (1938), 682-708.